Inequality 12

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Q> Let a, b, c be positive real numbers and $a+b+c=3$, then prove that,

$$\frac{a+1}{b^2+1} + \frac{b+1}{c^2+1} + \frac{c+1}{a^2+1} \geq 3$$

S

$$(b-1)^2 \geq 0$$
$$b^2 - 2b + 1 \geq 0$$
$$b^2 + 1 \geq 2b$$
$$\frac{1}{b^2+1} \leq \frac{1}{2b}$$
$$-\frac{1}{b^2+1} \geq -\frac{1}{2b}$$

Ans:- $$\frac{a+1}{b^2+1} = a+1 - \frac{b^2(a+1)}{b^2+1}$$

$$\geq a+1 - \frac{b^{\cancel{2}}(a+1)}{2\cancel{b}}$$

$$= a+1 - \frac{ab+b}{2}$$

$$\frac{b+1}{c^2+1} \geq b+1 - \frac{bc+c}{2}$$

$$\frac{c+1}{a^2+1} \geq c+1 - \frac{ca+a}{2}$$

$$S = a+b+c+3 - \frac{ab+b+bc+c+ca+a}{2}$$

$$= 3+3 - \frac{3+ab+bc+ca}{2}$$

$$= 3 + \frac{3}{2} - \frac{ab+bc+ca}{2} \geq 3$$

$$9 = (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$$

$$a \leq b \leq c$$
$$a^2+b^2+c^2 \geq ab+bc+ca$$

$$\Rightarrow 9 \leq 3(ab+bc+ca)$$
$$\Rightarrow ab+bc+ca \leq 3$$

Q> Let $a, b, c \in \mathbb{R}^+$, then prove that,

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

Ans :— HomeWork

Q) $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$ and $\sum_{i=1}^{n} a_i = 1$, then prove that,

$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{1-a_i}} \geq \frac{1}{\sqrt{n-1}} \sum_{i=1}^{n} \sqrt{a_i}$$

Ans :— Hint :- $\underbrace{\frac{a_i}{\sqrt{1-a_i}}}_{S} = \underbrace{\frac{1}{\sqrt{1-a_i}}}_{\geq \text{ something (say } S_1)} - \underbrace{\sqrt{1-a_i}}_{\leq \text{ something (say } S_2)}$

One approach to do so

$\Rightarrow S \geq S_1 - S_2$ but not to take always non-relative values

HomeWork

Q) $x_1, x_2, \ldots, x_n > 0$ such that $\frac{1}{1+x_1} + \cdots + \frac{1}{1+x_n} = 1$. Prove that,

$$x_1 x_2 \cdots x_n \geq (n-1)^n$$

Ans :— $y_i = \frac{1}{1+x_i} \Rightarrow x_i = \frac{1}{y_i} - 1$

$\Sigma y_i = 1$. To show $\prod_{i=1}^{n} \left(\frac{1}{y_i} - 1\right) \geq (n-1)^n$

HomeWork